

# **Helpful Resources for BC Calculus Summer Packet**

*Created and compiled by last year's BC  
Calculus students*

# Trigonometry

## Trigonometric Identities for AP Calculus

### Reciprocal identities

$$\sin u = \frac{1}{\csc u} \quad \cos u = \frac{1}{\sec u} \quad \tan u = \frac{1}{\cot u}$$

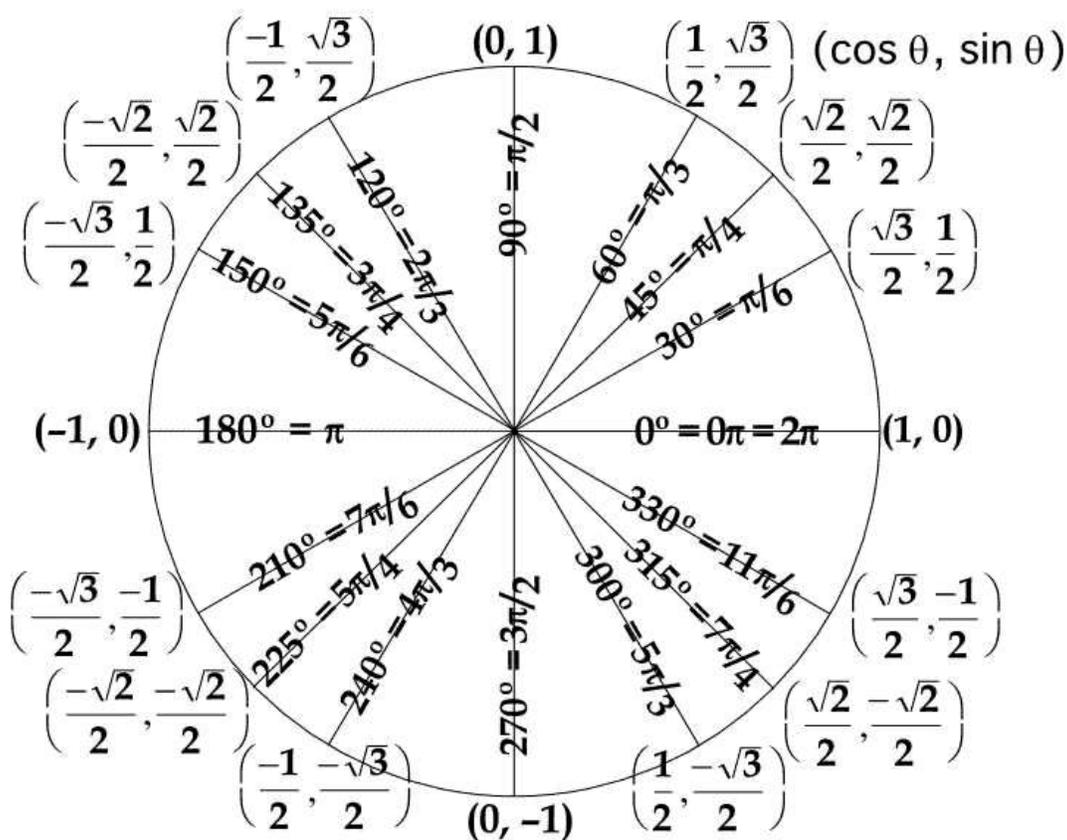
$$\csc u = \frac{1}{\sin u} \quad \sec u = \frac{1}{\cos u} \quad \cot u = \frac{1}{\tan u}$$

### Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1 \quad 1 + \tan^2 u = \sec^2 u \quad 1 + \cot^2 u = \csc^2 u$$

### Quotient Identities

$$\tan u = \frac{\sin u}{\cos u} \quad \cot u = \frac{\cos u}{\sin u}$$



Trigonometry tutorials:

<http://www.khanacademy.org/math/trigonometry/v/basic-trigonometry>

<http://www.khanacademy.org/math/trigonometry/v/basic-trigonometry-ii>

<http://www.khanacademy.org/math/trigonometry/v/radians-and-degrees>

<http://www.khanacademy.org/math/trigonometry/v/using-trig-functions>

<http://www.khanacademy.org/math/trigonometry/v/using-trig-functions-part-ii>

<http://www.khanacademy.org/math/trigonometry/v/the-unit-circle-definition-of-trigonometric-function>

<http://www.khanacademy.org/math/trigonometry/v/unit-circle-definition-of-trig-functions>

<http://www.khanacademy.org/math/trigonometry/v/determining-the-equation-of-a-trigonometric-function>

<http://www.khanacademy.org/math/trigonometry/v/trigonometric-identities>

<http://www.khanacademy.org/math/trigonometry/v/trig-identities-part-2--part-4-if-you-watch-the-proofs>

<http://www.khanacademy.org/math/trigonometry/v/trig-identities-part-3--part-5-if-you-watch-the-proofs>

# Type of Graphs

- <http://www.hippocampus.org/Calculus%20%26%20Advanced%20Math>

(click on “Calculus AB for AP” in the left hand column, then select the different topics from the right column.)

## Inverse Functions

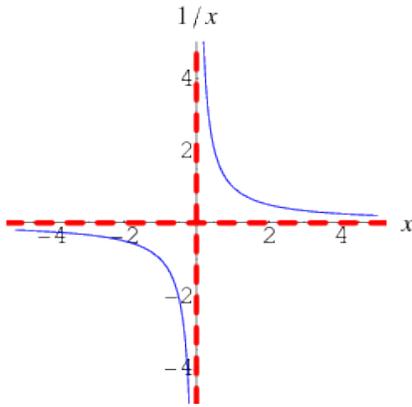
- <http://www.khanacademy.org/math/algebra-functions/v/introduction-to-function-inverses>

# Characteristics of Rational Functions

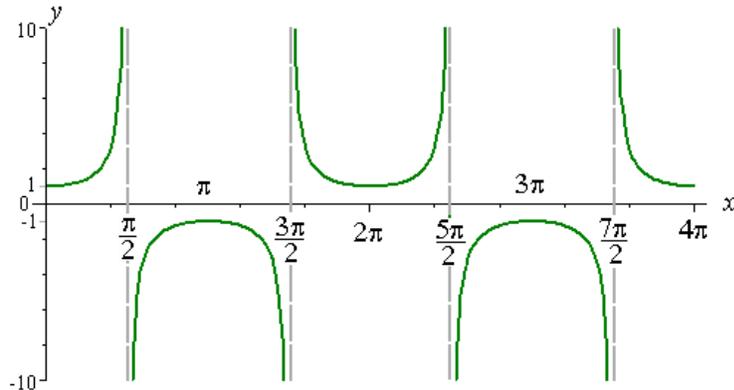
## Asymptotes

What are they?

An **asymptote** is a straight horizontal, vertical, or diagonal line that is approached when one of the variables in a given curve goes to infinity.



## Vertical Asymptote

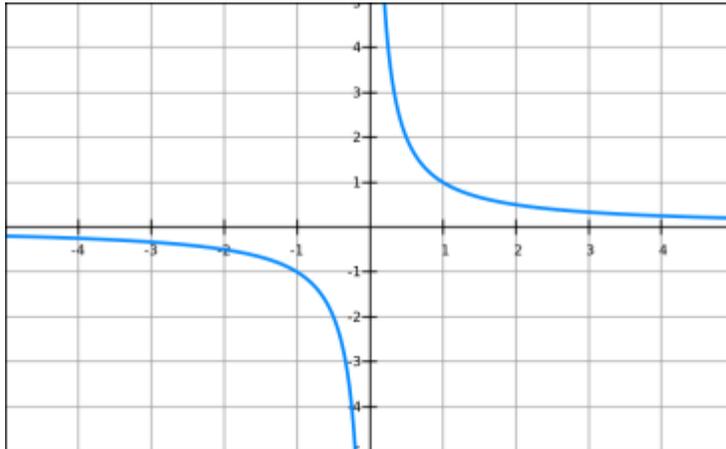


- Where  $y$  approaches infinity from both sides of an  $x$ -value
- Occur when the denominator of a fraction of a function equals 0
- A  $Y$ -value **cannot** exist at a point on the  $x$  axis that has an asymptote (in other words, the graph cannot touch the asymptote)

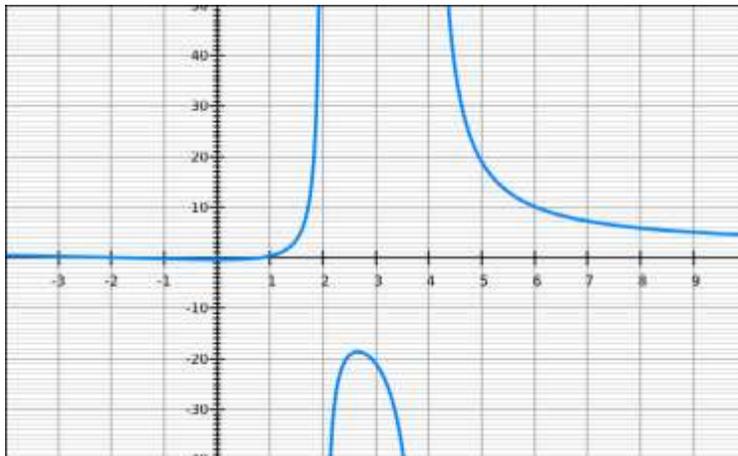
# How do you find a vertical asymptote?

[http://www.youtube.com/watch?v=qEOZNPce60&safety\\_mode=true&persist\\_safety\\_mode=1&safe=active](http://www.youtube.com/watch?v=qEOZNPce60&safety_mode=true&persist_safety_mode=1&safe=active)

When given an equation, such as  $y = \frac{1}{x}$  find the x-value(s) where the denominator equals 0. For this function, the denominator equals 0 when  $x=0$ . When you graph the equation, you will see that at  $x=0$   $y$  approaches infinity from both sides but does not ever exist at the point.



Now, given the function  $y = \frac{2x^2+2x-3}{x^2-6x+8}$  factor the denominator to figure out where it equals 0. Here, asymptotes occur at  $x=4$  and  $x=2$ , shown below



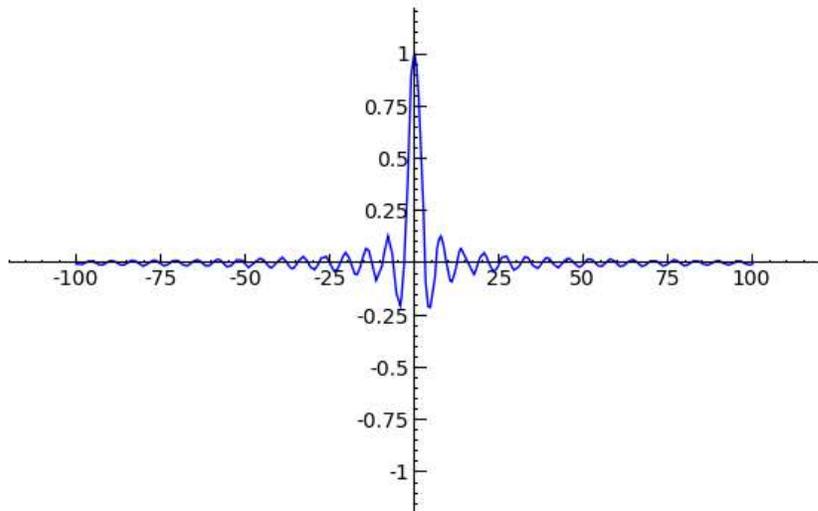
## Vertical asymptote vs hole

- A hole occurs at a point where an x value makes both the numerator and denominator 0
- To find where the hole is, factor the equation to its simplest form. The graph will take the shape of this new function, with a hole where the factored out part equals 0.

Example: In the equation  $y = \frac{x^2 - 4x + 4}{x + 2}$ , the numerator becomes  $(x+2)(x-2)$ . The two  $(x+2)$ 's cancel out, leaving the function as  $y = x - 2$ . Therefore, at  $x = 2$ , there will be a hole. To find out the  $y$  value, plug in  $x = 2$  into the new equation, to find that the hole is at  $(2, 0)$

## Horizontal Asymptote

A **Horizontal Asymptote** is the  $y$ -value that a graph approaches as  $x$  approaches positive or negative infinity



- The  $y$  value at which the graph “levels out” as  $x$  gets further from the origin
- A graph can touch or cross a horizontal asymptote, as shown above- the horizontal asymptotes are  $y = 0$ , despite the graph crossing the value many times.

## How do I find the horizontal asymptote?

[http://www.youtube.com/watch?v=c-yK2hUnSB0&safety\\_mode=true&persist\\_safety\\_mode=1&safe=active](http://www.youtube.com/watch?v=c-yK2hUnSB0&safety_mode=true&persist_safety_mode=1&safe=active)

To find the horizontal asymptote of a function as  $x$  approaches positive infinity, find the limit of the function. Using the rules for limits of rational functions the limit of  $y = \frac{2x^2 + 2x - 3}{x^2 - 6x + 8}$  is  $y = 2$ .

Therefore, as shown in the graph above, the graph gets closer and closer to  $y = 2$  as  $x$  approaches infinity.

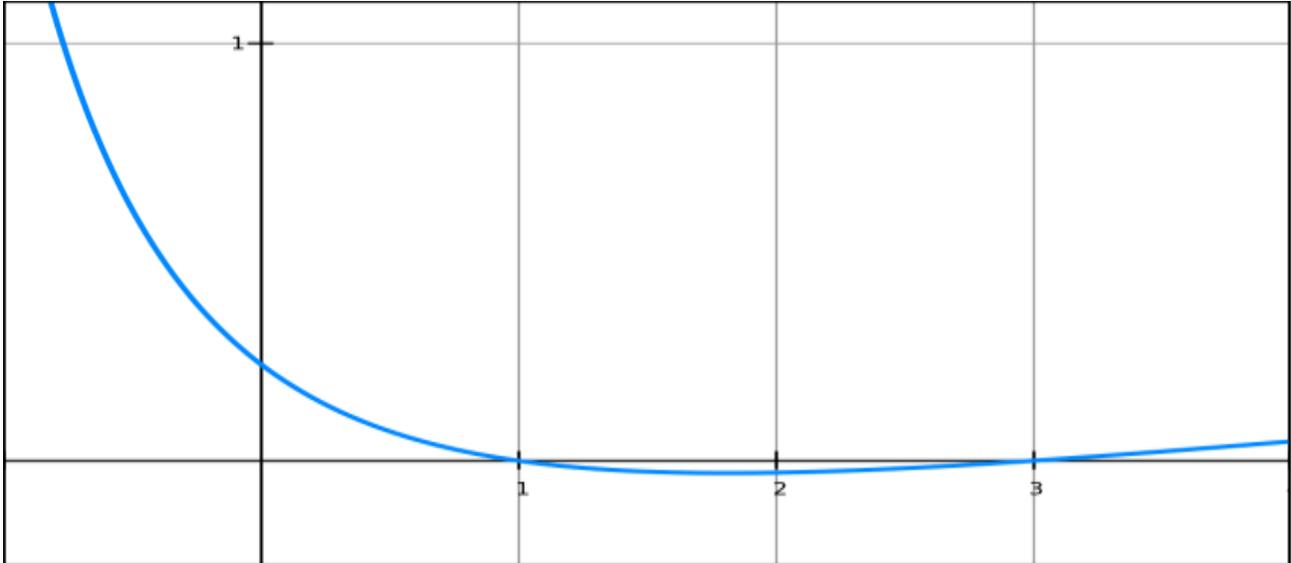
To find the horizontal asymptote of a function as  $x$  approaches negative infinity, plug in large negative numbers for  $x$  and see what the graph does. For the function,  $y = 5^x$ , as  $x$  becomes more and more negative, the function becomes a smaller and smaller function. Since it gets closer to 0, the horizontal asymptote as  $x$  approaches negative infinity is  $y = 0$ .

## Graph intersections

Finding where the graph intersects the  $x$ -axis:

- This is the point where  $y=0$ , since it intercepts the x axis. Therefore, set y equal to 0 in the function and solve for x.
- In a rational fraction function, this will mean that the x-intercepts are where the **numerator** equals 0.

Take the function  $y = \frac{x^2-4x+3}{x^2+9x+13}$ . When the numerator equals 0,  $x= 3$  and 1. At these x values, the graph intersects with the x-axis.



Finding where the graph intersects with the y-axis:

The graph intersects the y-axis when x equals 0. Therefore, set x equal to 0 in the function and solve for y. At this y value, the function intersects with the y-axis. For  $y = \frac{x^2-4x+3}{x^2+9x+13}$ , when  $x=0$ ,  $y=3/13$ . The function intersects the y axis at  $y=3/13$ .

## [Domain and Range \(click for video\)](#)

### Domain/Range with asymptotes

With a **vertical asymptote**, since the function can't cross it, the domain must reflect that the graph does not have a value at that x value. If a graph has a vertical asymptote of  $x=2$ , the domain would be

$$(-\infty, 2) \cup (2, \infty)$$

In a graph with a horizontal asymptote, the range consists of every y-value that exists in the

function. In  $y = \sqrt{\frac{1}{x}}$  since the horizontal asymptote is  $y=0$ , the range is  $(0, \infty)$

## Partial Fraction Decomposition

What we want to do in this section is to start with rational expressions and ask what simpler rational expressions did we add and/or subtract to get the original expression. The process of doing this is called **partial fractions** and the result is often called the **partial fraction decomposition**.

The process can be a little long and on occasion messy, but it is actually fairly simple. We will start by trying to determine the partial fraction decomposition of,

$$\frac{P(x)}{Q(x)}$$

where both  $P(x)$  and  $Q(x)$  are polynomials and the degree of  $P(x)$  is smaller than the degree of  $Q(x)$ . Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember.

**Example 1** Determine the partial fraction decomposition of the following.

$$\frac{8x - 42}{x^2 + 3x - 18}$$

### **Solution**

We'll go through the first one in great detail to show the complete partial fraction process and then we'll leave most of the explanation out of the remaining parts.

$$\frac{8x - 42}{x^2 + 3x - 18}$$

The first thing to do is factor the denominator as much as we can.

$$\frac{8x - 42}{x^2 + 3x - 18} = \frac{8x - 42}{(x + 6)(x - 3)}$$

So, by comparing to the table above it looks like the partial fraction decomposition must look like,

$$\frac{8x - 42}{x^2 + 3x - 18} = \frac{A}{x + 6} + \frac{B}{x - 3}$$

Note that we've got different coefficients for each term since there is no reason to think that they will be the same.

Now, we need to determine the values of  $A$  and  $B$ . The first step is to actually add the two terms back up. This is usually simpler than it might appear to be. Recall that we first need the least common denominator, but we've already got that from the original rational expression. In this case it is,

$$LCD = (x + 6)(x - 3)$$

Now, just look at each term and compare the denominator to the LCD. Multiply the numerator and denominator by whatever is missing then add. In this case this gives,

$$\frac{8x - 42}{x^2 + 3x - 18} = \frac{A(x - 3)}{(x + 6)(x - 3)} + \frac{B(x + 6)}{(x + 6)(x - 3)} = \frac{A(x - 3) + B(x + 6)}{(x + 6)(x - 3)}$$

We need values of  $A$  and  $B$  so that the numerator of the expression on the left is the same as the numerator of the term on the right. Or,

$$8x - 42 = A(x - 3) + B(x + 6)$$

This needs to be true regardless of the  $x$  that we plug into this equation. As noted above there are several ways to do this. One way will always work, but can be messy and will often require knowledge that we don't have yet. The other way will not always work, but when it does it will greatly reduce the amount of work required.

In this set of examples the second (and easier) method will always work so we'll be using that here. Here we are going to make use of the fact that this equation must be true regardless of the  $x$  that we plug in.

So let's pick an  $x$ , plug it in and see what happens. For no apparent reason let's try plugging in  $x = 3$ . Doing this gives,

$$\begin{aligned} 8(3) - 42 &= A(3 - 3) + B(3 + 6) \\ -18 &= 9B \\ -2 &= B \end{aligned}$$

Can you see why we choose this number? By choosing  $x = 3$  we got the term involving  $A$  to drop out and we were left with a simple equation that we can solve for  $B$ .

Now, we could also choose  $x = -6$  for exactly the same reason. Here is what happens if we use this value of  $x$ .

$$\begin{aligned} 8(-6) - 42 &= A(-6 - 3) + B(-6 + 6) \\ -90 &= -9A \\ 10 &= A \end{aligned}$$

So, by correctly picking  $x$  we were able to quickly and easily get the values of  $A$  and  $B$ . So, all that we need to do at this point is plug them in to finish the problem. Here is the partial fraction decomposition for this part.

$$\frac{8x - 42}{x^2 + 3x - 18} = \frac{10}{x + 6} + \frac{-2}{x - 3} = \frac{10}{x + 6} - \frac{2}{x - 3}$$

Notice that we moved the minus sign on the second term down to make the addition a subtraction. We will always do that.

# Properties of Exponents/ Exponentials and Logs

## Exponential and Logarithmic Functions:

$$\log_a x = b$$

$$a^b = x$$

$$\ln x = b$$

$$e^b = x$$

## Change of base formula:

$$\log_a x = \frac{\ln x}{\ln a}$$

## Laws of Logarithms:

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^r) = r \log_a x$$

## Laws of Exponents:

$$a^{x+y} = a^x a^y$$

$$a^{x-y} = \frac{a^x}{a^y}$$

$$(a^x)^y = a^{xy}$$

$$(ab)^x = a^x b^x$$

$$a^{-x} = \frac{1}{a^x}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$a^{1/x} = \sqrt[x]{a}$$

$$a^{x/y} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$$

$$\sqrt[x]{ab} = \sqrt[x]{a} \sqrt[x]{b}$$

$$\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$$

## Cancellation Equations:

$$\log_a(a^x) = x$$

$$a^{\log_a x} = x$$

$$\ln e^x = x$$

$$e^{\ln x} = x$$

<http://www.purplemath.com/modules/logs.htm>

# Limits

## BASIC VIDEO HELP

<http://mindbites.com/lesson/814-calculus-the-limit-laws-part-i>

[http://www.youtube.com/watch?v=qdm7cxAl-L0&safety\\_mode=true&persist\\_safety\\_mode=1&safe=active](http://www.youtube.com/watch?v=qdm7cxAl-L0&safety_mode=true&persist_safety_mode=1&safe=active)

## BASIC LIMIT DEFINITIONS AND LAWS

<http://mathworld.wolfram.com/Limit.html>

### What is a limit?

Consider the expression:

$$\lim_{x \rightarrow \infty} \frac{2x^2 + 1}{3x^2 + 4}$$

This is asking what value  $f(x)$  approaches as  $x$  gets very large. Try plugging in values:

$x$	$f(x)$
100	.6666111111
1,000	.6666661111
10,000	.6666666611

We can see that the value of  $f(x)$  approaches  $\frac{2}{3}$  as  $x$  gets large. So, the limit of  $f(x)$  as  $x$  approaches infinity is  $\frac{2}{3}$ .

### Evaluating limits as $x$ approaches infinity

To evaluate the following limits, suppose that  $a$  and  $b$  are constants such that  $a > b$  and  $c$  and  $d$  are random constants.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{cx^a}{dx^b} &= \infty \\ \lim_{x \rightarrow \infty} \frac{cx^a}{dx^a} &= \frac{c}{d} \\ \lim_{x \rightarrow \infty} \frac{cx^b}{dx^b} &= \frac{c}{d} \\ \lim_{x \rightarrow \infty} \frac{cx^b}{dx^a} &= 0\end{aligned}$$

When evaluating limits, only consider the terms of highest degree. For example:

$$\lim_{x \rightarrow \infty} \frac{x^3 + x}{x^2 + 1}$$

Since  $x^3$  is the highest term in the numerator and  $x^2$  is the highest term in the denominator, evaluate the limit as such:

$$\lim_{x \rightarrow \infty} \frac{x^3 + x}{x^2 + 1} = \lim_{x \rightarrow \infty} \frac{x^3}{x^2} = \infty$$

### **Evaluating limits as $x$ approaches a set value**

To evaluate a limit in which  $x$  approaches a set value, simply plug that value into the equation. For example:

$$\lim_{x \rightarrow 5} \frac{x^3 + x}{x^2 + 1} = \frac{(5)^3 + (5)}{(5)^2 + 1} = \frac{125 + 5}{26} = 5$$

For some functions, finding the limit at a set number will yield an undefined value or an indeterminate value (i.e.  $\frac{\infty}{\infty}$  or  $\frac{0}{0}$ ). For example:

$$\lim_{x \rightarrow 0} \frac{x^3}{x^2}$$

would appear to produce an answer of 0. In this case, first simplify the expression and then substitute the value. So:

$$\lim_{x \rightarrow 0} \frac{x^3}{x^2} = \lim_{x \rightarrow 0} x = 0$$

- <http://www.calculus-help.com/tutorials/>
- <http://www.khanacademy.org/math/calculus/v/introduction-to-limits>
- <http://www.mathsisfun.com/calculus/limits-infinity.html>